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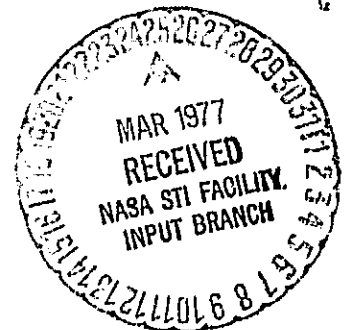
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ALGORITHM FOR ESTIMATING MIXTURE PROPORTIONS  
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BY JAMES SPARRA  
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A Stochastic Approximation Algorithm for  
Estimating Mixture Proportions

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1. Summary. A stochastic approximation algorithm for estimating the proportions in a mixture of normal densities is presented. The algorithm is shown to converge to the true proportions in the case of a mixture of two normal densities.

2. Introduction. Let  $\Lambda = \{\alpha \in R^m: \alpha_i > 0 \text{ and } \sum_{i=1}^m \alpha_i = 1\}$ . For each  $i$ ,  $i = 1, \dots, m$ , let  $\mu_i$  be an element of  $R^n$  and  $\Sigma_i$  be a positive definite real symmetric  $n \times n$  matrix. Let  $X$  be a random variable with values in  $R^n$  and with density function.

$$p(\hat{\alpha}, x) = \sum_{i=1}^m \hat{\alpha}_i p_i(x), \text{ for } x \in R^n$$

where  $\hat{\alpha} \in \Lambda$  and

$$p_i(x) = (2\pi)^{-n/2} |\Sigma_i|^{-1/2} \exp\left\{-\frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1}(x-\mu_i)\right\}$$

for each  $i = 1, \dots, m$ .

We assume that  $\hat{\alpha}$  is not known but that  $\mu_i$  and  $\Sigma_i$  are known for  $i = 1, \dots, m$ . An algorithm for estimating  $\hat{\alpha}$  will be presented in part 3 of this paper and in part 4 the algorithm will be shown to converge to  $\hat{\alpha}$  in mean square and with probability 1 in the case where  $m = 2$ .

3. The Algorithm. Let  $\{x_k\}_{k=0}^{\infty}$  be a sequence of observations on  $X$ . Let  $\alpha^0 \in A$ . For  $n \geq 0$  define  $\alpha^{n+1}$  by

$$\alpha_i^{n+1} = \alpha_i^n - c_n \left( \alpha_i^n - \frac{\alpha_i^n p_i(x_n)}{p_{\alpha^n}(x_n)} \right),$$

where

$$p_{\alpha^n}(x_n) = \sum_{i=1}^m \alpha_i^n p_i(x_n)$$

and  $\{c_k\}_{k=0}^{\infty}$  is a sequence of positive numbers such that

$$\sum_{k=0}^{\infty} c_k = \infty \quad \text{and} \quad \sum_{k=0}^{\infty} c_k^2 < \infty.$$

We note that each iterate is in  $A$  and that, since  $X$  is a random variable, each iterate may itself be considered a random variable.

#### 4. Convergence of the Algorithm.

Theorem: If  $\hat{\alpha} \in R^2$  then the algorithm described in part 3 converges to  $\hat{\alpha}$  in mean square and with probability 1.

Proof: We refer the reader to the algorithm described in [1, pp. 332-333] and to the proof of convergence given in [1, pp. 350-352]. The applicability of the theorem given there is clear if we let  $f(\alpha) = E(Z_{\alpha})$ , for each  $\alpha \in A$ , where

$$(Z_{\alpha})_i = \alpha_i - \frac{\alpha_i (p_i \circ X)}{p_{\alpha} \circ X}.$$

In order to show convergence we must show that conditions (A1)-(A3) in [1, pp. 332-333] are satisfied. First we note that

$$f(\alpha) = (\alpha_1 - \alpha_1 g_1(\alpha_1), \alpha_2 - \alpha_2 g_2(\alpha_2))$$

where

$$g_1(\alpha_1) = \int_{R^n} \frac{p_1(x)}{\alpha_1 p_1(x) + (1-\alpha_1)p_2(x)} p_{\hat{\alpha}}(x) dx$$

and

$$g_2(\alpha_2) = \int_{R^n} \frac{p_2(x)}{(1-\alpha_2)p_1(x) + \alpha_2 p_2(x)} p_{\hat{\alpha}}(x) dx.$$

Further, we note that

$$\frac{d^2 g_1(\alpha_1)}{d\alpha_1^2} = \int_{R^n} \frac{p_1(x) [p_1(x) - p_2(x)]^2}{[\alpha_1 p_1(x) + (1-\alpha_1)p_2(x)]^3} \cdot p_{\hat{\alpha}}(x) dx > 0$$

and

$$\frac{d^2 g_2(\alpha_2)}{d\alpha_2^2} = \int_{R^n} \frac{p_2(x) [p_2(x) - p_1(x)]^2}{[(1-\alpha_2)p_1(x) + \alpha_2 p_2(x)]^3} \cdot p_{\hat{\alpha}}(x) dx > 0.$$

Now,  $g_1(\hat{\alpha}_1) = 1$  and  $g_1(1) = 1$ . So, since  $g_1$  has positive second derivative we have that  $g_1(\alpha_1) < 1$  if  $\alpha_1 \in (\hat{\alpha}_1, 1)$  and  $g_1(\alpha_1) > 1$  if  $\alpha_1 \in (0, \hat{\alpha}_1)$ .

Similarly,  $g_2(\hat{\alpha}_2) = 1$  and  $g_2(1) = 1$  and  $g_2(\alpha_2) < 1$  if  $\alpha_2 \in (\hat{\alpha}_2, 1)$  and  $g_2(\alpha_2) > 1$  if  $\alpha_2 \in (0, \hat{\alpha}_2)$ .

We now show that (A1)-(A3) are satisfied: Let  $\alpha \in \Lambda$ . Then

$$(A1) \quad f(\alpha) = 0 \quad \text{iff} \quad g_1(\alpha_1) = 1 = g_2(\alpha_2) \quad \text{iff} \quad \alpha = \hat{\alpha}.$$

$$(A2) \quad (\alpha - \hat{\alpha})^T f(\alpha) = (\alpha_1 - \hat{\alpha}_1)(\alpha_1 - \alpha_1 g_1(\alpha_1)) + (\alpha_2 - \hat{\alpha}_2)(\alpha_2 - \alpha_2 g_2(\alpha_2)).$$

If  $\alpha_1 > \hat{\alpha}_1$  then  $g_1(\alpha_1) < 1$  and  $(\alpha_1 - \alpha_1 g_1(\alpha_1)) > 0$ . Then also

$\alpha_2 < \hat{\alpha}_2$  and  $g_2(\alpha_2) > 1$  and  $(\alpha_2 - \alpha_2 g_2(\alpha_2)) < 0$ . Thus, if

$\alpha_1 > \hat{\alpha}_1$  then  $(\alpha - \hat{\alpha})^T f(\alpha) > 0$ . Similarly, if  $\alpha_1 < \hat{\alpha}_1$  then

$(\alpha - \hat{\alpha})^T f(\alpha) > 0$ . Thus, A2 is satisfied in any closed, convex

subset of  $\Lambda$ .

(A3)

$$E(\|Z_\alpha\|^2) = \sum_{i=1}^2 (\alpha_i^2 - 2 \int_{\mathbb{R}^n} \frac{\alpha_i^2 p_i(x)}{p_\alpha(x)} \cdot p_\alpha(x) dx + \int_{\mathbb{R}^n} \left( \frac{\alpha_i p_i(x)}{p_\alpha(x)} \right)^2 \cdot p_\alpha(x) dx)$$

Now, we note that each term in the  $i$ th summand,  $i = 1, 2$ , is

less than 1 so that there is an  $h > 0$  such that  $E(\|Z_\alpha\|^2) < h$

for all  $\alpha \in \Lambda$  and A3 is satisfied.

## Bibliography

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